

# Time evolution of Hamiltonian constraint system: an idea applicable to quantum gravity

*Junichi Iwasaki*

iwasaki@physicist.net

Oct 26, 2012

## Abstract

The Hamiltonian constraint system is the canonical formulation of a physical system with a Hamiltonian constrained to vanish. In terms of the canonical variables, we define what we call reference observable, with respect to which other observables evolve. We study if it plays the role of time. As simple examples, we study the theories of non-relativistic and relativistic particles. We outline an application of the idea to general relativity.

## 1 Introduction

The Hamiltonian constraint system is the canonical formulation of a physical system with a Hamiltonian constrained to vanish. This Hamiltonian generates the evolution of observables with respect to a parameter given in the theory. This parameter is arbitrarily reparametrizable and hence it is not physical time but a gauge parameter. Physical time is given as an observable in some theories or not given explicitly in the other theories. In terms of canonical variables, we define what we call reference observable, with respect to which other observables evolve. We study if the reference observable plays the role of time.

General relativity (GR) is a Hamiltonian constraint system. The time coordinate given in GR is not physical time. Physical time is not explicitly given in the theory. Therefore, a reference observable, if defined, is perhaps utilized as physical time. The idea of reference observable already exists although not clearly realized. It is in the theory of relativistic particle (RP), a Lorentz covariant theory, which is another Hamiltonian constraint system much simpler than GR. RP is a trajectory theory while GR is a field theory. Its physical time is one of the observables although it is not Lorentz invariant. Lorentz invariant time is not given explicitly in the theory. Furthermore, The theory of non-relativistic particle (NP), a Galilei covariant theory, is known to be written as a Hamiltonian constraint system, another trajectory theory simpler than RP. Its physical time is one of the observables and Galilei invariant.

NP is a toy we use to study the idea in Sec.2. We see the reference observable we define for NP is the physical time, which is Galilei invariant. Upon quantization, we see a Heisenberg picture with respect to the reference observable is available.

We extend the idea to RP in Sec.3. We see the reference observable we define for RP is the proper time, which is Lorentz invariant. We also see that a Heisenberg picture with respect to the reference observable can be constructed in quantization.

The definition of the reference observable is not particular to the theories studied here but general enough for the Hamiltonian constraint system. We outline an application of the idea to GR in Sec.4. We conclude in Sec.5.

The present work is motivated by the use of the Heisenberg picture rather than the Schrödinger picture [1] and relational spacetime rather than absolute spacetime [2] in quantization of non-perturbative canonical gravity [3].

## 2 Non-relativistic particle (NP)

### 2.1 Gauge independent formulation

NP or more precisely the non-relativistic trajectory of a free particle is often described as a Hamiltonian system without constraint. The action is

$$S_0 := \int \left( p \frac{dx}{dt} - H_0 \right) dt, \quad (1)$$

where  $x$  and  $p$  are the position and momentum of the particle respectively,  $t$  is physical time and

$$H_0 := \frac{p^2}{2m} \quad (2)$$

is the Hamiltonian.  $m$  is the mass of the particle. Here the space is assumed to be one dimensional for simplicity.

$x$  and  $p$  are the canonical variables of the Hamiltonian system and satisfy the Poisson bracket relations  $\{x, p\} = 1$  and  $\{x, x\} = \{p, p\} = 0$ . They are functions of  $t$  and their evolution in  $t$  is generated by the Hamiltonian.

The equations of motion are

$$\frac{dx}{dt} = \{x, H_0\} = \frac{p}{m}, \quad (3)$$

$$\frac{dp}{dt} = \{p, H_0\} = 0. \quad (4)$$

This system has no constraint nor gauge parameter. The equations of motion above present physics. We refer this system to gauge independent formulation and the equations of motion above to gauge independent equations of motion.

Quantization is a process of replacing  $p$  by  $\hat{p} := -i \frac{\partial}{\partial x}$  and the Poisson bracket by  $(-i)$  times the commutator in a representation space in which  $x$  is diagonal. The state function is a function of  $x$  and  $t$ . By applying the time evolution operator to a Fourier mode, the physical state with momentum  $p$  is

$$\psi_0(x, t) = e^{-i \frac{p^2}{2m} t} e^{ipx} = e^{i \left( px - \frac{p^2}{2m} t \right)} \quad (5)$$

$$= e^{i \int_0^p \left( x - \frac{k}{m} t \right) dk}. \quad (6)$$

Here,  $x$  is the position of the particle at time  $t$  and, when moving with momentum  $p$ ,  $x - \frac{p}{m}t$  is the position at  $t = 0$ . Therefore, (6) is in the Heisenberg picture while (5) is in the Schrödinger picture. Since the classical solutions are labelled by the initial position and momentum, the physical state in the Heisenberg picture is seen as a function of the classical solutions.

The gauge independent formulation for NP summarized in this subsection is well understood and a good toy for us to study the idea introduced in this paper.

## 2.2 Gauge covariant formulation

NP is restructured as a Hamiltonian constraint system. The action is

$$S_N := \int \left( p \frac{dx}{d\tau} + e \frac{dt}{d\tau} - H_N \right) d\tau, \quad (7)$$

where  $\tau$  is an arbitrary parameter and

$$H_N := \lambda \left( e + \frac{p^2}{2m} \right) \quad (8)$$

is the Hamiltonian.  $\lambda$  is a multiplier to force  $e = -\frac{p^2}{2m}$ . Physical time  $t$  is treated as another canonical variable with its canonical conjugate  $e$  satisfying  $\{t, e\} = 1$  and  $\{t, t\} = \{e, e\} = 0$ . The canonical variables are now functions of  $\tau$  and their evolution in  $\tau$  is generated by the Hamiltonian, which is constrained to vanish.

Note that by substituting  $e = -\frac{p^2}{2m}$ ,  $S_0$  is restored from  $S_N$  provided  $t$  is a monotonic function of  $\tau$ . However, without doing so, we restore physical content of the gauge independent formulation in Sec.2.3.

The equations of motion together with the constraint are

$$\dot{t} = \{t, H_N\} = \lambda, \quad (9)$$

$$\dot{x} = \{x, H_N\} = \lambda \frac{p}{m}, \quad (10)$$

$$\dot{e} = \{e, H_N\} = 0, \quad (11)$$

$$\dot{p} = \{p, H_N\} = 0, \quad (12)$$

$$e + \frac{p^2}{2m} = 0. \quad (13)$$

Here the dot implies the derivative with respect to  $\tau$ . In the phase space spanned by the canonical variables, we refer the subspace satisfying the constraint (13) to the constraint surface. Because  $\tau$  is arbitrarily reparametrizable, the equations of motion above are gauge dependent. Rather, they are gauge covariant since the change of  $\lambda$  together with a reparametrization of  $\tau$  results in the same equations in form. We refer them to gauge covariant equations of motion. Accordingly, we refer the Hamiltonian constraint system to gauge covariant formulation. As we see in the following sections, RP and GR are in this kind of systems.

## 2.3 Galilei invariant reference observable

In this subsection, we define what we call reference observable. Then, using it, we derive the gauge independent equations of motion from the gauge covariant ones.

We denote the reference observable by  $s$ . The definition is such that it is an observable satisfying  $\dot{s} = \lambda$  on the constraint surface.  $t$  satisfies this condition and hence can be a reference observable. However, we look for a more general form of  $s$  so that the idea can be applied to RP and GR, that is,  $s$  containing terms of higher order in canonical variable. It can be

$$s := t - c \left( \frac{te}{m} + \frac{xp}{2m} \right), \quad (14)$$

where  $c$  is a constant, so that

$$\dot{s} = \{s, H_N\} = \lambda - \frac{c}{m} H_N. \quad (15)$$

The second term in (15) vanishes on the constraint surface. This means that the second term in (14) is constant in  $\tau$  on the constraint surface.

In the gauge covariant formulation, the canonical variables are functions of  $\tau$ . Here, we restrict ourselves to a sector in which the canonical variables are functions of  $s$ , through which they implicitly depend on  $\tau$ , that is,  $t(\tau) := t(s(\tau))$ ,  $x(\tau) := x(s(\tau))$ ,  $e(\tau) := e(s(\tau))$  and  $p(\tau) := p(s(\tau))$ . This sector contains all the solutions for the gauge independent equations of motion. If  $\lambda > 0$ , then the sector is identical to the space of the canonical variables we started with.

We use  $\frac{d}{d\tau} = \frac{ds}{d\tau} \frac{d}{ds} = \lambda \frac{d}{ds}$  to rewrite the gauge covariant equations of motion as follows.

$$\frac{dt}{ds} = \{t, e + H_0\} = 1, \quad (16)$$

$$\frac{dx}{ds} = \{x, e + H_0\} = \{x, H_0\} = \frac{p}{m}, \quad (17)$$

$$\frac{de}{ds} = \{e, e + H_0\} = \{e, H_0\} = 0, \quad (18)$$

$$\frac{dp}{ds} = \{p, e + H_0\} = \{p, H_0\} = 0. \quad (19)$$

These equations and their solutions are independent of  $\tau$ . Since equation (16) implies  $s = t$  up to a constant term and equation (19) implies (18) on the constraint surface, the gauge independent equations of motion (3) and (4) are restored.

We see the reference observable is in fact the physical time of the theory and Galilei invariant. It must be pointed out that the definition of  $s$  here is not particular to the present theory but general enough for the Hamiltonian constraint system.

## 2.4 Heisenberg picture in reference observable

In the gauge covariant formulation, quantization is a process of replacing  $p$  and  $e$  by  $\hat{p} := -i \frac{\partial}{\partial x}$  and  $\hat{e} := -i \frac{\partial}{\partial t}$  respectively and the Poisson bracket by  $(-i)$  times the

commutator in a representation space in which  $x$  and  $t$  are diagonal. Accordingly, the equations of motion and the constraint are replaced by the respective representation as operators on the state function.

The state function is a function of  $x$ ,  $t$  and  $s$ . However, the physical state should not depend on  $s$  because it must be equal to  $\psi_0$ , that is gauge independent. The physical state with momentum  $p$  is

$$\psi_N(x, t) = e^{-i\left(\hat{e} + \frac{\hat{p}^2}{2m}\right)s} e^{i(px+et)} = e^{i\left(px+et - \left(e + \frac{p^2}{2m}\right)s\right)} \quad (20)$$

$$= e^{i\int_0^p \left(x - \frac{k}{m}s\right)dk} e^{i\int_0^e (t-s)dh} \quad (21)$$

with  $e + \frac{p^2}{2m} = 0$ . Here, the operator applied makes an evolution in  $s$ .  $x$  and  $t$  are the position and time of the particle respectively at the reference observable  $s$  and, when moving with momentum  $p$ ,  $x - \frac{p}{m}s$  and  $t - s$  are the position and time respectively at  $s = 0$ . Therefore, with respect to the reference observable, (21) is in the Heisenberg picture while (20) is in the Schrödinger picture before imposing the constraint. However, after imposing the constraint, (20) cannot be in the Schrödinger picture since the  $s$  dependence in (20) disappears as expected. Nevertheless, the Heisenberg picture of (21) holds because the presence of  $s$  in (21) is not due to the state function itself but due to the gauge dependence of  $x$  and  $t$ . Hence, the reference observable  $s$  behaves as if it is time in the Heisenberg picture.

### 3 Relativistic particle (RP)

#### 3.1 Gauge covariant formulation

RP or more precisely the relativistic trajectory of a free particle is often described as a Hamiltonian constraint system, which is a gauge covariant formulation in our classification. The action is

$$S_R := \int \left( p_\mu \frac{dx^\mu}{d\tau} - H_R \right) d\tau, \quad (22)$$

where  $x^\mu$  and  $p_\mu$  are the 4-position and 4-momentum of the particle respectively,  $\tau$  is an arbitrary parameter and

$$H_R := \frac{\lambda}{2m} (p_\mu p^\mu + m^2) \quad (23)$$

is the Hamiltonian.  $\lambda$  is a multiplier to force  $p_\mu p^\mu + m^2 = 0$  and  $m$  is the mass of the particle. Here the Greek letter indices run 0 through 3 and are raised or lowered by Minkowski metric with the signature  $(-1, +1, +1, +1)$  and the sum over repeated indices is understood unless otherwise explicitly stated.

$x^\mu$  and  $p_\mu$  are the canonical variables of the theory and satisfy the Poisson bracket relations  $\{x^\mu, p_\nu\} = \delta^\mu_\nu$  and  $\{x^\mu, x^\nu\} = \{p_\mu, p_\nu\} = 0$ . They are observables of the

theory. They are functions of  $\tau$  and their evolution in  $\tau$  is generated by the Hamiltonian, which is constrained to vanish. The gauge covariant equations of motion together with the constraint are

$$\dot{x}^\mu = \{x^\mu, H_R\} = \frac{\lambda}{m} p^\mu, \quad (24)$$

$$\dot{p}_\mu = \{p_\mu, H_R\} = 0, \quad (25)$$

$$p_\mu p^\mu + m^2 = 0, \quad (26)$$

where the dot implies the derivative with respect to  $\tau$ . We refer the subspace in the phase space satisfying the constraint (26) to the constraint surface.

### 3.2 Lorentz invariant reference observable

In this subsection, we define a reference observable. Then, we derive the equations of motion with respect to it, that is, the gauge independent equations of motion.

We denote the reference observable by  $s$ . The definition is such that it is an observable satisfying  $\dot{s} = \lambda$  on the constraint surface. It is satisfied by

$$s := -\frac{x^\mu p_\mu}{m}, \quad (27)$$

up to a constant term. It is straightforward to show that

$$\dot{s} = \{s, H_R\} = \lambda - \frac{2}{m} H_R. \quad (28)$$

In the gauge covariant formulation, the canonical variables are functions of  $\tau$ . Here, we restrict ourselves to a sector in which the canonical variables are functions of  $s$ , through which they implicitly depend on  $\tau$ , that is,  $x^\mu(\tau) := x^\mu(s(\tau))$ , and  $p_\mu(\tau) := p_\mu(s(\tau))$ . This sector contains all the solutions for the gauge independent equations of motion. If  $\lambda > 0$ , then the sector is identical to the space of the canonical variables we started with.

We use  $\frac{d}{d\tau} = \frac{ds}{d\tau} \frac{d}{ds} = \lambda \frac{d}{ds}$ , to rewrite the gauge covariant equations of motion as follows.

$$\frac{dx^\mu}{ds} = \left\{ x^\mu, \frac{p_\nu p^\nu + m^2}{2m} \right\} = \left\{ x^\mu, \frac{p_\nu p^\nu}{2m} \right\} = \frac{p^\mu}{m}, \quad (29)$$

$$\frac{dp_\mu}{ds} = \left\{ p_\mu, \frac{p_\nu p^\nu + m^2}{2m} \right\} = \left\{ p_\mu, \frac{p_\nu p^\nu}{2m} \right\} = 0. \quad (30)$$

These equations are independent of  $\tau$ .

The reference observable  $s$  for the present theory is known to be the proper time since  $\sqrt{-\frac{dx_\mu}{ds} \frac{dx^\mu}{ds}} = \sqrt{-\frac{p_\mu}{m} \frac{p^\mu}{m}} = 1$ , although this definition of the proper time is not commonly discussed. It must be pointed out that the definition of  $s$  here is not particular to the present theory but general enough for the Hamiltonian constraint system.

### 3.3 Lorentz invariant Heisenberg picture

Quantization is a process of replacing  $p_\mu$  by  $\hat{p}_\mu := -i\frac{\partial}{\partial x^\mu}$  and the Poisson bracket by  $(-i)$  times the commutator in a representation space in which  $x^\mu$  is diagonal.

The state function is a function of  $x^\mu$  and  $s$ . The physical state with momentum  $p_\mu$  is

$$\psi_R(x) = e^{-i\frac{\hat{p}_\nu \hat{p}^\nu + m^2}{2m}s} e^{ip_\mu x^\mu} = e^{i\left(p_\mu x^\mu - \frac{p_\nu p^\nu + m^2}{2m}s\right)} \quad (31)$$

$$= e^{i\int_0^P \left(x^\mu - \frac{k^\mu}{m}s\right) dk_\mu} e^{-i\frac{m}{2}s} \quad (32)$$

with  $p_\mu p^\mu + m^2 = 0$ . Here, the operator applied makes an evolution in  $s$ .  $x^\mu$  is the 4-position of the particle at the reference observable  $s$  and, when moving with 4-momentum  $p_\mu$ ,  $x^\mu - \frac{p^\mu}{m}s$  is the 4-position at  $s = 0$ . Therefore, with respect to the reference observable, (32) is in the Heisenberg picture while (31) is in the Schrödinger picture before imposing the constraint. However, after imposing the constraint, (31) cannot be in the Schrödinger picture since the  $s$  dependence in (31) disappears. Nevertheless, the Heisenberg picture of (32) holds because the presence of  $s$  in (32) is not due to the state function itself but due to the gauge dependence of  $x^\mu$ . Hence, the reference observable  $s$  behaves as if it is a time in the Heisenberg picture. The overall phase factor  $e^{-i\frac{m}{2}s}$  does not affect physics.

## 4 General relativity (GR)

### 4.1 Gauge covariant formulation

The idea developed in the previous sections for NP and RP is applied to GR. The three theories are Hamiltonian constraint systems, that is, gauge covariant formulations in our classification. The main difference of GR from NP and RP is that GR is a field theory while NP and RP are trajectory theories. The gauge parameter for GR is at least 4 dimensional while it is one dimensional for NP or RP.

The action is

$$S_G := \int \left( \int P(x) \frac{dQ(x)}{dt} d^3x - H_G \right) dt, \quad (33)$$

where  $Q$  and  $P$  are the canonical variables, whose physical meaning and number of indices depend on the formulation of GR. Here, we suppress possible indices from the canonical variables for simplicity of notation. The integrals are over the spatial part and time of spacetime coordinates  $x^\mu = (t, x^a)$ , and

$$H_G := \int N^\mu C_\mu d^3x. \quad (34)$$

is the Hamiltonian.  $N^0$  and  $N^a (a = 1, 2, 3)$  are respectively the lapse and shift functions.  $C_0 = 0$  and  $C_a = 0$  are known as Hamiltonian and diffeomorphism constraints

respectively. The coordinates  $x^\mu$  do not present physical spacetime but are arbitrary coordinates.

The canonical variables,  $Q$  and  $P$ , satisfy the Poisson bracket relations  $\{Q(x), P(y)\} = \delta^3(x, y)$  and  $\{Q(x), Q(y)\} = \{P(x), P(y)\} = 0$ . They are functions of  $x^\mu$  and their evolution in  $t$  is generated by the Hamiltonian  $H_G$ , which is constrained to vanish. In addition, their diffeomorphisms in  $x^a$  are generated by the diffeomorphism constraints  $C_a$  in the Hamiltonian.

The gauge covariant equations of motion and the constraints are formally

$$\frac{df(x)}{dt} = \int d^3y N^\mu(y) \{f(x), C_\mu(y)\}, \quad (35)$$

$$C^\mu(x) = 0, \quad (36)$$

where  $f(x)$  is  $Q(x)$  or  $P(x)$ . We refer the subspace in the phase space satisfying the constraints (36) to the constraint surface.

## 4.2 Spacetime reference observables

In this subsection, we define reference observables. Then, we derive the equations of motion with respect to them, that is, what we think gauge independent equations of motion although they are still formal here.

We define reference observables and denote them by  $s^\mu$ . They are observables satisfying  $\frac{ds^\mu}{dt} = N^\mu$  on the constraint surface.

The canonical variables are functions of  $x^\mu$ . We restrict ourselves to a sector in which the canonical variables are functions of  $s^\mu$ , through which they implicitly depend on  $x^\mu$ , that is,  $f(x) := f(s(x))$ . If  $|\frac{\partial s^\mu}{\partial x^\nu}| > 0$ , then the sector is identical to the space of the canonical variables we started with.

We use  $\frac{d}{dt} = \frac{ds^\mu}{dt} \frac{\partial}{\partial s^\mu} = N^\mu \frac{\partial}{\partial s^\mu}$  or more explicitly

$$\frac{df(x)}{dt} = \frac{ds^\mu(x)}{dt} \frac{\partial f(x)}{\partial s^\mu} = \int d^3y N^\mu(y) \delta^3(y, x) \frac{\partial f(x)}{\partial s^\mu} \quad (37)$$

to rewrite the gauge covariant equations of motion as follows.

$$\frac{\partial f(x)}{\partial s^\mu} = \int d^3y \{f(x), C_\mu(y)\}. \quad (38)$$

Here, we have used the fact that  $N^\mu$  is arbitrary.

The reference observables for NP and RP were known as physical time and the proper time respectively. However, we do not know what the reference observables  $s^\mu$  for GR are. Details are left for future work.

## 4.3 Spacetime Heisenberg picture

Quantization is a process of replacing  $P(x)$  by  $\hat{P}(x) := -i\frac{\delta}{\delta Q(x)}$  and the Poisson bracket by  $(-i)$  times the commutator in a representation space in which  $Q(x)$  is diagonal.



The state function is a functional of  $Q$  and  $s^\mu$ . The physical state suggested by NP and RP is

$$\Psi_G[Q] = e^{-i \int \hat{C}_\mu(x) s^\mu(x) d^3x} e^{i \int P(x) Q(x) d^3x} = e^{i \int (P(x) Q(x) - C_\mu(x) s^\mu(x)) d^3x} \quad (39)$$

$$= e^{i \int d^3x \int_0^{P(x)} \left( Q(x) - \int d^3y \frac{\delta C_\mu(y)}{\delta P(x)} s^\mu(y) \right) dP(x)} \quad (40)$$

with  $C_\mu = 0$ . Here, the operator applied makes an evolution in  $s^\mu$ .  $Q(x)$  is the value of  $Q$  at the reference observables  $s^\mu(x)$  and, when propagating with  $P(x)$ ,  $Q(x) - \int d^3y \frac{\delta C_\mu(y)}{\delta P(x)} s^\mu(y)$  is the value of  $Q$  at the reference observables  $s^\mu(x) = 0$ . Therefore, with respect to the reference observables, (40) is in the Heisenberg picture while (39) is in the Schrödinger picture before imposing the constraints. However, after imposing the constraints, (39) cannot be in the Schrödinger picture since the  $s^\mu$  dependence in (39) disappears. Nevertheless, the Heisenberg picture of (40) holds because the presence of  $s^\mu$  in (40) is not due to the state function itself but due to the gauge dependence of  $Q(x)$ . Perhaps, the reference observables  $s^\mu$  might behave as if they are spacetime in the Heisenberg picture.

## 5 Conclusion

We have studied the Hamiltonian constraint system using simple models (NP and RP). In order to study the time evolution, we defined reference observables, with respect to which other observables evolve. The definition was general enough for the Hamiltonian constraint system, including GR.

First, we studied NP, which already had physical time. We derived equations of motion with respect to the reference observable. It was known that the theory was well described in terms of the physical time. Then, we understood that the reference observable we defined for NP was in fact the physical time. Upon quantization, we constructed a Heisenberg picture with respect to the reference observable.

Next, we studied RP, which did not have Lorentz invariant time, without defining one. We derived equations of motion with respect to the reference observable. It was known that the theory was well described in terms of the proper time. Then, we understood that the reference observable we defined for RP was in fact the proper time. Upon quantization, we constructed a Lorentz invariant Heisenberg picture with respect to the reference observable.

Finally, we outlined an application of the idea to GR, which had no time. Its time coordinate was not physical time. For GR, a field theory, a set of reference observables were defined. We derived equations of motion with respect to the set of reference observables. Upon quantization, we constructed a Heisenberg picture with respect to them. However, we did not know what the reference observables we defined for GR were. Details were left for future work.

## References

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